Laminar Boundary Layer on a Cone at Incidence in Supersonic Flow

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Results of three-dimensional boundary-layer calculations for a pointed cone and comparisons with experimental data are presented. Using a similarity transformation, the explicit dependence of the equations on axial position is removed. A further transformation is made to remove a singularity at the leeward symmetry plane. The resulting equations, in two independent variables, are then solved with an implicit finite difference technique by marching around the cone from the windward to the leeward symmetry plane. A technique for computing the edge properties from an experimental pressure distribution is developed. The comparisons with experimental data demonstrate that the theory provides an adequate representation of the viscous flow over most of the cone surface, although the physical model breaks down near the leeward symmetry plane.

Nomenclature

c = Chapman-Rubesin factor
c_v = heat capacity at constant volume
F = nondimensional radial velocity, u/u_e
G = nondimensional cross-flow velocity, w/w_e
h = enthalpy, height within boundary layer
H = nondimensional enthalpy, h/h_e
k = three-dimensionality parameter
l = $\rho\mu$ product, [Eq. (6)]
M = Mach number
p = pressure
r = radial distance from apex of cone
S = entropy
$\Delta S_1, \Delta S_2$ = coefficients tabulated by Sims (Ref. 28)
u = radial velocity
v = velocity normal to surface, or in θ direction
$\mathbf{v}$ = unit vector in freestream direction
V = variable defined by Eq. (5)
w = cross-flow velocity (in ξ direction)
XYZ = Cartesian frame in Fig. 1
y = coordinate normal to surface
z = logarithmic variable defined by Eq. (7)
α = angle of attack
β = angle between limiting streamline and cone generator
θ = spherical polar angle measured from cone axis
θ_c = semivertex angle of cone
λ = similarity coordinate [Eq. (1)]
μ = viscosity
ξ = circumferential coordinate $\xi = \phi \sin\theta_c$ (angle in developed surface of cone)
ρ = density
σ = Prandtl number
ϕ = circumferential angle measured from windward meridian
ψ = angle from leeward meridian in developed surface [Eq. (8)]

Subscripts

c = cone
e = edge
r = reference
w = wall
∞ = freestream

Superscript

$'$ = denotes differentiation with respect to ξ

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I. Introduction

THE laminar boundary layer on a cone at incidence is of practical importance in several applications, such as high-speed aircraft and lifting re-entry vehicles. For lifting re-entry in particular, a knowledge of the full three-dimensional boundary-layer properties is essential for accurately estimating the local heat transfer and skin friction, including the determination of separated flow regions. The computational technique described here is shown to provide a reasonably accurate prediction of the viscous flow for circular cones at angles of attack less than the semivertex angle.

The majority of the literature on viscous conical flow has been concerned with approximate methods. Integral methods have been published by Yen and Thyson[1] and by Chang et al,[2] both of which showed reasonable agreement with heat transfer data. Vaglio-Laurin and Hoffert[3] have presented a two-region integral method, which includes effects of strong viscous interaction. DaForno[4] has developed a technique for cones of arbitrary cross section, based on the cold-wall approximation of Vaglio-Laurin,[5] which is valid for small cross flow. Another small cross-flow approach has been published by Pinkus and Cousin.[6] They apply a technique developed by Cooke[7] for using a three-dimensional "equivalent radius" in existing axisymmetric boundary-layer programs to produce a corresponding solution for a yawed cone. The method is limited to small angles of attack. Recently, Chan[8] has published a comparison of the small cross-flow theory of Tsen and Arnandon[9] with experimental data, showing remarkably good agreement even at fairly large angles of attack.

Moore[10] first showed in 1951 that the absence of radial gradients in the inviscid conical flow makes possible a similarity transformation equivalent to the Blasuis transformation for a flat plate. He subsequently published a solution[11] to the transformed equations in the limit of very small angle of attack, corresponding to Stone's[12,13] perturbation solution for the inviscid flow. In the present paper, Moore's transformation has been used to reduce the equations to two independent variables. A further transformation is then made to remove a singularity at the leeward symmetry plane. Without assuming small cross-flow, the reduced equations are solved with an implicit finite difference technique by marching from the windward to the leeward symmetry plane. Similar methods have been reported by Vvedenskaya[14] and by Cooke.[15] As a verification of the present application, the heat-transfer calculations of Vvedenskaya were reproduced quite closely. However, differences between the present results and those of Cooke are noted in Ref. 16.

The objective of this work is to determine the validity of the approach by comparisons of the theory with test data. These comparisons include heat transfer, pitot tube measurements, and separation line location. A technique is developed for using the experimental pressure distribution to obtain the other edge properties. Using this technique, experimental heat-transfer measurements are matched to within 6%. The influence of other treatments of the edge properties is also investigated.

Interesting results are also obtained which show that in most instances a solution to the laminar boundary-layer equations does not exist at the leeward symmetry plane even where the region-of-influence principle is not violated.

II. Governing Equations and Boundary Conditions

The three-dimensional boundary-layer equations for a cone have been derived by Moore.[10] He introduced the similarity transformation

$$\lambda = c\left(\frac{3u_r}{rp}\right)^{1/2}\int_0^y \rho dy \tag{1}$$

in order to obtain the following set of equations in two independent variables:

r momentum;

$$w_eG\frac{\partial F}{\partial \xi} = V\frac{\partial F}{\partial \lambda} + \frac{w_e^2}{u_e}G^2 - \frac{u_e'w_e}{u_e}FG + 3\frac{\partial}{\partial \lambda}\left(\ell\frac{\partial F}{\partial \lambda}\right) \tag{2}$$

ϕ momentum;

$$w_eG\frac{\partial G}{\partial \xi} = V\frac{\partial G}{\partial \lambda} - u_eFG - w_e'G^2 - \frac{p'}{\rho w_e} + 3\frac{\partial}{\partial \lambda}\left(\ell\frac{\partial G}{\partial \lambda}\right) \tag{3}$$

energy;

$$w_eG\frac{\partial H}{\partial \xi} = V\frac{\partial H}{\partial \lambda} + \frac{w_eGp'}{\rho h_e} - \frac{w_eGh_e'}{h_e}H + \frac{3\ell}{h_e}\left[u_e^2\left(\frac{\partial F}{\partial \lambda}\right)^2 + w_e^2\left(\frac{\partial G}{\partial \lambda}\right)^2\right] + 3\frac{\partial}{\partial \lambda}\left(\frac{\ell}{\sigma}\frac{\partial H}{\partial \lambda}\right) \tag{4}$$

continuity;

$$\partial V/\partial \lambda = \tfrac{3}{2}u_eF + (w_e\partial G/\partial \xi) + w_e'G + (w_e/2)Gp'/p \tag{5}$$

where

$$\ell = c^2\mu\rho/p \tag{6}$$

In these equations, F and G are the radial and cross-flow velocity components, normalized by their respective edge values u_e and w_e. H is the enthalpy normalized by its edge value h_e and p and ρ are the pressure and density, respectively. The function V defined by Eq. (5) has no physical interpretation. The coordinate $\xi = \phi \sin\theta_c$ is the angle from the windward meridian in the developed surface of the cone (Fig. 1).

Examination of Eqs. (2–4) shows that they are parabolic partial differential equations in which ξ is the time-like direction. The equations are therefore to be solved starting from given initial values at $\xi = 0$ and marching in ξ. At each step, the change in the profiles can be obtained by solving Eqs. (2–4) for the partial derivatives $\partial F/\partial\xi$, $\partial G/\partial\xi$, and $\partial H/\partial\xi$, using an appropriate finite difference scheme to advance the solution. However, at the two symmetry planes ($\xi = 0$ and $\xi = \pi \sin\theta_c$) the cross-flow velocity component w_e vanishes due to symmetry. Since w_e is a coefficient of the ξ derivatives in all three equations, the equations have a singularity at both symmetry planes. No difficulty is encountered in integrating away from the windward symmetry plane, $\xi = 0$. Both the viscous and inviscid flow are well-defined there, so that no difficulty is expected. However, at the leeward symmetry plane, a complex interaction between the inviscid flow and the boundary layer occurs. Numerical difficulties are anticipated there in view of Moore's problems with obtaining a solution on the leeward symmetry plane.[17] To better condition the equations for numerical calculations in this region, the singularity is transformed to $z = \infty$ by replacing ξ with the variable;

$$z = -\ln[1 - (\xi/\pi \sin\theta_c)] \tag{7}$$

With this change, the left-hand sides of Eqs. (2–4) are each of the form $(w_e/\psi)G\partial\Phi/\partial z$, where Φ represents either F, G, or H, and the angle ψ is defined by

$$\psi = (\pi - \phi)\sin\theta_c \tag{8}$$

At the leeward plane of symmetry, $\xi = \pi \sin\theta_c$ and

$$\lim_{\xi\to\pi \sin\theta_c}(w_e/\psi) = [-w_e']_{\xi = \pi\sin\theta_c}$$

Since w_e' is nonzero at the leeward ray, the singularity in the equations has now been eliminated. However, in order to obtain the solution at the leeward ray, it is now necessary to integrate to $z \to \infty$. As the integration is carried to large values of z, all edge conditions approach constant values. The solution at the leeward ray, if it exists, is then obtained as the asymptotic "steady-state" solution.

On the windward plane of symmetry, Eqs. (2–5) reduce to the following set of ordinary differential equations which can be integrated to obtain initial profiles for the subsequent marching procedure:

$$3\frac{d}{d\lambda}\left(\ell\frac{dF}{d\lambda}\right) + V\frac{dF}{d\lambda} = 0 \tag{9}$$

$$3\frac{d}{d\lambda}\left(\ell\frac{dG}{d\lambda}\right) + V\frac{dG}{d\lambda} - u_eFG - w_e'G^2 - \frac{p''}{\rho w_e'} = 0 \tag{10}$$

$$3\frac{d}{d\lambda}\left(\frac{\ell}{\sigma}\frac{dH}{d\lambda}\right) + V\frac{dH}{d\lambda} + \frac{3\ell}{h_e}\left[u_e\frac{dF}{d\lambda}\right]^2 = 0 \tag{11}$$

$$dV/d\lambda = \tfrac{3}{2}u_eF + w_e'G \tag{12}$$

The boundary conditions required for Eqs. (2–5) or Eqs. (9–12) are as follows: At $\lambda = 0$, $F = u/u_e = 0$, $G = w/w_e = 0$, and either $H = h/h_e = h_w(\xi)/h_e$ for the case of specified wall enthalpy, or $\partial H/\partial\lambda = 0$, for the case of zero heat transfer. At the outer edge of the boundary layer, the velocities approach the wall conditions of the outer inviscid flow which, together with the pressure distribution, $p(\xi)$, are presumed known. Thus for $\lambda \to \infty$, $F \to 1$, $G \to 1$, and $H \to 1$. The appropriate boundary condition at the wall for Eq. (5), allowing for the possibility of blowing or suction, is

$$V(0,\xi) = -c(3ru_r/p)^{1/2}(\rho v)_{r_1}$$

where $(\rho v)_{r_1}$ is the mass transfer at a given axial location r_1, which may be a function of circumferential location. In order to maintain the similarity form of the equation, it is necessary to assume that $(\rho v)_w$ varies as $r^{-1/2}$. For the case of specified wall temperature, the heat transfer to the wall is also proportional to $r^{-1/2}$. A reasonable simulation of the ablation process is therefore possible, since the ablation rate will generally be approximately proportional to the heat transfer. No provisions are made for the differences in gas properties of the injectant or for chemical reactions. Thus, only the kinematics of blowing can be studied.

III. Numerical Methods

Initial profiles for the marching procedure are obtained by solving Eqs. (9–12), which are a system of second-order ordinary differential equations with split boundary conditions. A variety of methods have been developed for this type of problem. The technique used in obtaining the present results is

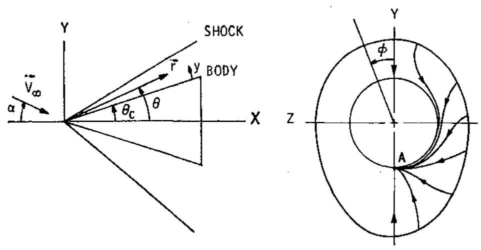

Fig. 1 Coordinate systems and cross-flow streamline pattern in shock layer for cone at angle of attack.

termed the "method of accelerated successive nonlinear displacements" and was first proposed by Lieberstein.[18] The convergence of the method was studied by Bryan[19] and applications to boundary-layer problems have been reported previously by Lew[20] and by Strom.[21] The method consists of writing the differential equations in finite difference form and obtaining successive improvements to the initially assumed solution by a Newton-Raphson type procedure applied at each point.[16]

The implicit marching procedure used to solve Eqs. (2–5) was originally devised by Flügge-Lotz and Blottner,[22] and has since been applied successfully to both two-dimensional[23–25] and three-dimensional flows.[26] The general approach is to satisfy the linearized difference form of the equations midway between the new step and the previous one. Because of the non-linearity of the equations, iteration of the solution is used to reduce the error in the initial estimate of the coefficients. When the difference equations for the unknown at the new step are written in matrix form, a tridiagonal set of linear algebraic equations results. That is, the nth equation contains unknowns only at the n-1, n, and $n + 1$ locations through the boundary layer. The tridiagonal form makes possible an efficient scheme for solving the simultaneous equations.[16]

IV. Edge Conditions

The calculation of the boundary layer requires that the inviscid edge conditions be prescribed. The calculation of these inviscid properties is complicated by the formation of a steep entropy gradient near the body and a singularity which occurs at the leeward symmetry plane, due to the convergence of the cross-flow streamlines shown in Fig. 1. Therefore, several methods for obtaining the inviscid edge properties were investigated. These include the numerical conical flow solution by Moretti,[27] the perturbation solution by Stone[12,13] as tabulated by Sims[28] and a method proposed by Cooke.[15]

The simplest approach is to use Stone's perturbation solution for the inviscid flow, which ignores the entropy layer. Sims[28] has observed that although the entropy on the body must be constant, the perturbation expression,

$$\Delta S/c_v = \Delta S_1 + \alpha \Delta S_2 \cos\phi \qquad (13)$$

is an adequate definition of the entropy outside the vortical layer (to first order in angle of attack). The edge conditions tabulated by Sims are therefore applicable when the boundary layer is thicker than the entropy layer, but still thin compared with the shock-layer thickness. However, the boundary-layer equations just developed are not strictly valid in this region, because the edge conditions are not independent of r as required by the similarity transformation. This r dependency results from the different growth rates of the boundary layer (which grows as $r^{1/2}$) and the entropy layer (which is conical and thus grows in proportion to r). Nevertheless, one can assume that local similarity applies so that Sim's results can be used to obtain an indication of the effect of a circumferential variation in edge entropy on the boundary-layer behavior.

Alternatively, one can accept Sim's perturbation solution for the radial velocity as being correct, and compute the remaining properties from it, together with the assumption of constant entropy at the surface. The cross-flow velocity w_e is obtained from the inviscid r momentum equation in spherical coordinates, which on the surface of the cone reduces to

$$w_e(u_e' - w_e) = 0 \qquad (14)$$

Equation 14 shows that w_e can be obtained by differentiating the radial velocity (provided $w_e \neq 0$). The surface pressure can then be obtained from the two velocity components and the entropy by using the conservation of total enthalpy (which can be derived as an integral of the inviscid cross-flow momentum equation). This procedure was used by Cooke.[15] However, at large Mach numbers, the resulting pressure distribution is considerably in error.

Both of the edge treatments outlined in the preceding paragraph rely on the first-order perturbation solution for the velocity components. Considerable inaccuracy can therefore be expected with these methods at moderate to large angles of attack. Recently, numerical methods[27,29–31] have been developed for computing the inviscid conical flow. These methods are applicable at relative angles of attack α/θ_c as large as unity. For this study, a modified version of the Moretti[27] program was used to obtain the majority of the results. The numerically computed pressure distribution is generally a good approximation to the experimental data. However, the computed values of the velocity components, u and v, do not generally satisfy Eq. (14) on the body surface. Furthermore, the conical flow solutions tabulated by Babenko[30] suffer from the same difficulty at the larger angles of attack. To ensure a consistent set of edge conditions, therefore, only the pressure distribution and the entropy on the windward symmetry plane from the Moretti program were used. The remaining variables were recomputed by the method discussed below.

On the surface of the cone, the inviscid cross-flow momentum equation reduces to

$$w_e[w_e' + u_e] + p'/\rho_e = 0 \qquad (15)$$

The pressure distribution and the body entropy at the surface are taken from the Moretti program. Equation (15) is then numerically integrated to obtain w_e. The density ρ_e at each point is found from the pressure and the constant entropy condition. The radial velocity u_e is obtained from the pressure, density, and cross-flow velocity by using the conservation of total enthalpy. Equation (15) has a singularity at points where the cross-flow w_e vanishes. These singularities cause some difficulties in the numerical integration, as discussed in Ref. 16. In general, however, it is possible to obtain a valid solution within the region where the boundary-layer equations are valid.

The integration of Eq. (15) can also be carried out using an experimental pressure distribution, with the entropy com-

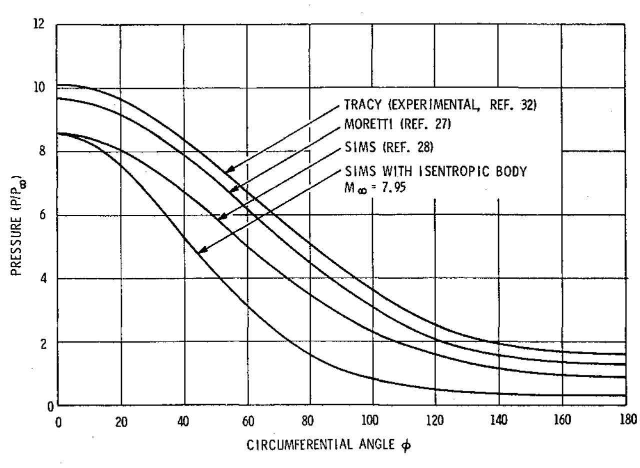

Fig. 2 Comparison of pressure distribution on 10° cone by various methods, $\alpha = 8°$.

puted from the shock angle in the windward plane of symmetry. With the edge conditions thus determined directly from experimental data, a more precise evaluation of the boundary-layer theory is possible.

A comparison of the pressure distributions calculated by the various methods previously discussed is given in Fig. 2. The comparisons are made for a 10° cone at Mach 7.95 at an angle of attack of 8°. These conditions permit comparison with the test data of Tracy.[32] The pressure distribution from the Moretti inviscid conical flow program gives the best agreement with the data. An estimate of the induced pressure has been shown[16] to account for part of the difference between theory and experiment. The first-order perturbation solution of Sims also gives a fairly good approximation to the data. As explained earlier, Sims' solution applies outside the entropy layer, where the first order entropy variation is given by Eq. (13). If the entropy at the body is held constant as was done by Cooke,[15] the pressure distribution is substantially altered as shown in Fig. 2. However, the effect is much smaller at lower Mach numbers.

V. Results

Difficulties at the Leeward Plane of Symmetry

Moore[17] encountered numerical difficulties when attempting to solve Eqs. (9–12) on the leeward symmetry plane. In an attempt to understand the reasons for the difficulty, he obtained an asymptotic solution of the cross-flow momentum equation, Eq. (10), under the special assumptions of adiabatic wall, Prandtl number of unity, and linear temperature-viscosity relation. This asymptotic solution is unique provided that the quantity k, defined by

$$k = -(\tfrac{2}{3})w_e'/u_e$$

is greater than $-\frac{1}{3}$. For $k < -\frac{1}{3}$, an additional asymptotic solution is possible, and another undetermined constant appears.[17] Moore conjectured that this indeterminacy was due to the lack of previous history for the fluid which enters the region of influence of the leeward symmetry plane from around the cone. He pointed out that for $k = -\frac{1}{3}$ the inviscid streamlines near $\phi = \pi$ are parabolas with focus at the cone apex, and are thus tangent to the parabolic region of influence. For k more negative than $-\frac{1}{3}$, the inviscid streamlines move inward relative to the region of influence of the symmetry plane. When this happens, the solution is no longer uniquely determined by the inviscid properties in the symmetry plane alone, but must also depend upon the fluid which has been carried around the cone. Because the streamlines at the base of the boundary layer are more steeply inclined toward the leeward plane than the inviscid streamlines, Moore suggested that the correct criteria for uniqueness should actually be

$$k > -\tfrac{1}{3}\lim_{\lambda\to 0}(F/G) = -\tfrac{1}{3}[(\partial F/\partial\lambda)/(\partial G/\partial\lambda)]_{\lambda=0} \quad (16)$$

Table 1 Results obtained in testing Moore's indeterminacy hypothesis

Case	$[k]_{\phi=\pi}$	$\left[-\frac{1}{3}\frac{\partial F/\partial\lambda}{\partial G/\partial\lambda}\right]_{\phi=\phi}$	Leeward plane solution obtained
$\theta_c = 40°$, $\alpha = 10°$, $M_\infty = 5$, cool wall	−0.166	−0.156	Yes
$\theta_c = 20°$, $\alpha = 5°$, $M_\infty = 5$, cool wall	−0.152	−0.091	Yes
$\theta_c = 7.5°$, $\alpha = 1.5°$, $M_\infty = 3.1$, adiabatic wall	−0.218	...	No
$\theta_c = 7.5°$, $\alpha = 2°$, $M_\infty = 3.1$, adiabatic wall	−0.298	...	No

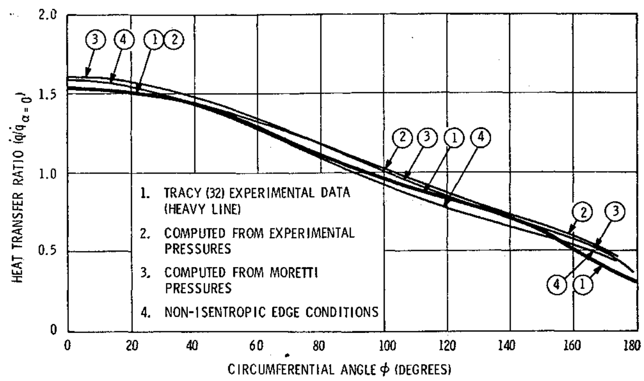

Fig. 3 Heat-transfer distribution on 10 cone at Mach 7.95 and $\alpha = 4°$.

If this hypothesis were correct, then it should be possible to obtain a solution on the leeward ray by integrating around the cone from the windward to the leeward plane of symmetry. Because the previous boundary-layer history is contained in such a calculation, the leeward ray solution should be obtainable regardless of whether $k > -\frac{1}{3}$. In particular, it should be possible to obtain leeward plane of symmetry solutions by this method, which are not obtainable from the ordinary differential equations.

Several cases were calculated to test the aforementioned conjectures, the results of which are summarized in Table 1. As the first two cases show, it is possible to compute leeward ray solutions for k more negative than the criteria of Eq. (16). However, solutions for $k < -\frac{1}{3}$, or even near $-\frac{1}{3}$ could not be obtained as shown by the last two cases in Table 1. Furthermore, for the two cases in Table 1 where the leeward plane solution was obtained, it also proved possible to obtain the same solution from the ordinary differential equations. Thus, it appears that the calculation of the boundary layer over the entire cone is no better than any other method for obtaining solutions on the leeward plane of symmetry. Furthermore, it is clear that the calculation of previous fluid history is not a sufficient condition to guarantee a solution to the boundary-layer equations. The fact that a solution does not always exist at the leeward plane has interesting implications for the calculation of three-dimensional boundary-layer flow over a blunt cone. Presumably, this type of calculation should also break down as the flow asymptotically becomes conical.

Comparison with Heat-Transfer Data

Figures 3–4 show a comparison of the heat transfer predicted from the laminar boundary-layer theory (using various edge conditions as previously described) with Tracy's[32]

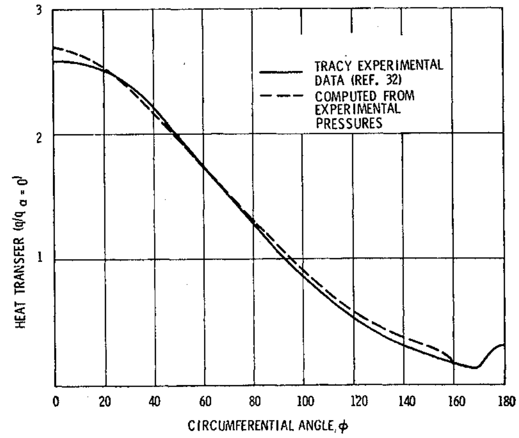

Fig. 4 Heat-transfer distribution on 10 cone at Mach 7.95 and $\alpha = 12°$.

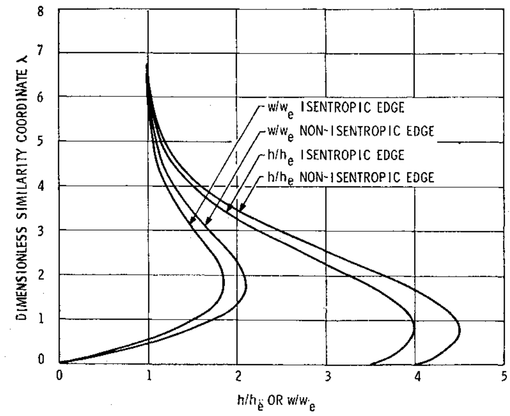

Fig. 5 Comparison of boundary-layer profiles computed with isentropic and nonisentropic edge conditions.

test data. Using the experimental pressure distributions, the experimental heat-transfer results are predicted within 6%, except near the leeward meridian. The theoretical pressure distributions, although somewhat lower than measured, also give a satisfactory prediction of heat transfer. Surprisingly, the use of Sims' nonisentropic edge properties does not greatly affect the boundary-layer structure. This is further illustrated in Fig. 5, which shows typical boundary-layer profiles for Tracy's 10° cones at $\phi = 117°$, as computed with both the Moretti pressure distribution and with the nonisentropic values from Sims. Most of the difference in the heat transfer predicted from these two sources of edge properties arises from Sims' lower edge pressures. As shown in Fig. 5, there is very little difference in the slope of the enthalpy profiles.

Figure 6 compares the predicted axial variation in heat transfer (based on a pressure distribution from Moretti's program) with experimental data from Ref. 33. Here the results have not been normalized by $\dot{q}_{\alpha=0}$ as with the Tracy data. The agreement is quite good, although some scatter in the data is evident. These data were taken near transition which may account for some of the scatter. The predicted variation of heat transfer as $r^{-1/2}$ is generally supported by the data.

Boundary-Layer Thickness

Figure 7 shows a comparison of experimental and theoretical boundary-layer thicknesses. The experimental thicknesses were determined from Tracy's pitot probe data.[32] Figure 8 shows that the agreement with the predicted boundary-layer thickness is generally good, up to about 150°. At that point,

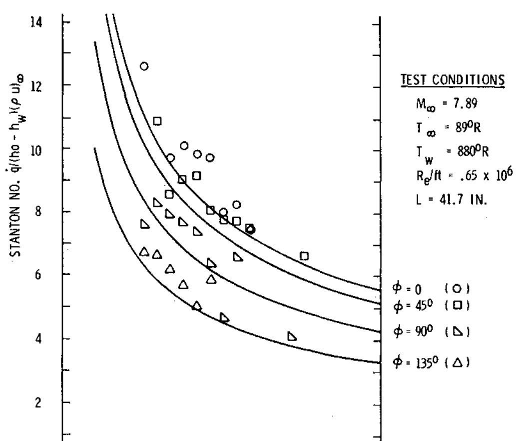

Fig. 6 Comparison of theory with heat-transfer data for 7.25° cone, $\alpha = 2.5°$, $M_\infty = 7.89$ (Ref. 33).

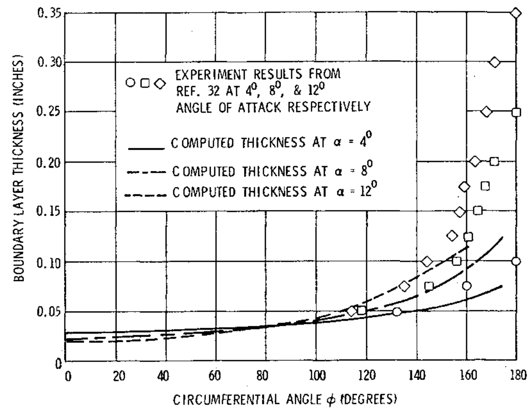

Fig. 7 Boundary-layer thickness for 10° cone at Mach 7.95 and various angles of attack.

the experimental boundary layer undergoes a sharp increase in thickness, which is not matched by the theoretical calculation. Instead, the theoretical thicknesses (for $\alpha = 4°$ and $8°$) approach $\phi = \pi$ at a definite nonzero slope; evidence of the discontinuous derivatives there discussed by Vvedenskaya.[14] At $\alpha = 12°$, the boundary-layer calculations indicate separation at 160°. At $\alpha = 8°$, the boundary layer is very close to being separated (an adverse pressure gradient exists), so the sharp boundary layer growth may be evidence of separation or incipient separation in that case also. However, the boundary layer is certainly not separated at $\alpha = 4°$, and yet there is a clearly identifiable increase in the experimental boundary-layer thickness, quite similar to that at $\alpha = 8°$ and $\alpha = 12°$. Therefore, the increase in thickness may not be due to separation alone, but to a change in structure required to avoid the discontinuous derivatives predicted by theory.

Comparison with Pitot Data

The Pitot probe readings predicted from the boundary-layer calculations are compared with Tracy's experimental results at $\alpha = 8°$ in Fig. 8. The comparison is generally satisfactory, although the decrease in predicted probe pressure should be somewhat steeper. There is a noticeable difference between the predicted and experimental probe readings outside the boundary layer. The difference is due to the entropy gradient of the inviscid flow. The theoretical calculations were made with a constant edge entropy equal to the value at the windward ray. In fact, however, this streamline is entrained by the boundary layer and the actual edge entropy is lower, giving a higher pitot reading at the edge of the boundary layer.

Prediction of Upwash Angle and Separation

The predicted upwash angle on the surface of a 10° cone at $\alpha = 4°$ is compared in Fig. 9 with flow visualization data from

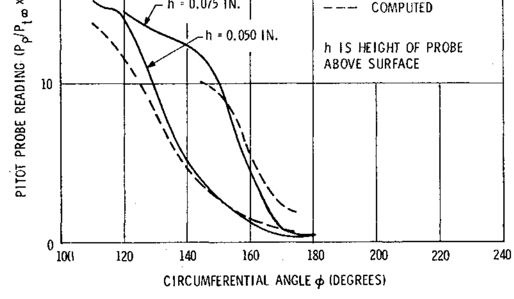

Fig. 8 Pitot probe profiles through boundary layer on 10° cone at Mach 7.95 and $\alpha = 8°$.

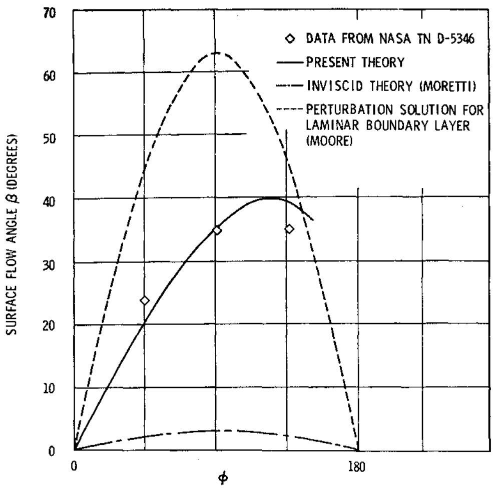

Fig. 9 Comparison of theoretical upwash angles for a 10° cone at $\alpha = 4°$, $M_\infty = 14.6$ (helium) with data from Ref. 34 obtained with oil-film technique, adiabatic wall.

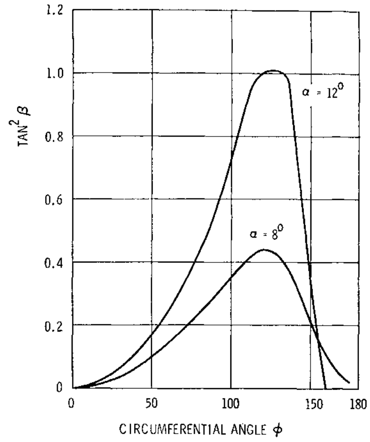

Fig. 10 $\tan^2\beta$ vs ϕ for 10° cone at Mach 7.95 at 8° and 12° angle of attack.

Reference 34. The agreement is good and generally within experimental error. The theoretical boundary-layer calculations were based on a pressure distribution from Moretti's program. The upwash angle of the flow at the surface is substantially higher than predicted by inviscid theory because the low-velocity fluid in the boundary layer is more easily turned by the circumferential pressure gradient. The improvement of the present solution over the perturbation theory of Moore[11] is also apparent in Fig. 9. Even at this relatively small angle of attack, the peak cross-flow velocity is more than 20% of the radial component, which is beyond the range of validity of Moore's theory.

The boundary-layer theory remains valid within the attached flow region for arbitrarily large angles of attack, provided the external flow is conical. However, as the separation line is approached, the number of iterations required at each step, to achieve a specified accuracy, is increased. In order to keep the number of iterations within reason, the step size must be made smaller and smaller. The calculation cannot be extended into the reverse flow region. This behavior was first observed by Cooke,[15] who also noted that a theoretical analysis of the incompressible boundary layer predicted a linear behavior for $\tan^2\beta$ vs ϕ near the separation point (where β is the angle between a cone generator and the limiting streamline at the wall). He found that $\tan^2\beta$ was in fact nearly linear near separation for the compressible case as well. Accordingly, a plot of $\tan^2\beta$ for the 8° and 12° angle- of-attack cases of Tracy[32] is shown in Fig. 10. In both cases, the experimental pressure distribution was used to obtain the results. The calculations at $\alpha = 12°$ clearly predicts separation at 160°. However, the experimental heat-transfer distribution (Fig. 4) indicates that the separation does not occur until 170°. This small difference may be due to breakdown of the physical model at separation, or in the interpretation of the heat-transfer data, since the minimum heat transfer does not necessarily coincide with the separation line. The friction heating, for example, depends on the sum of the radial and circumferential shear terms. Since the radial shear is positive and decreasing with ϕ at separation, the minimum frictional heating will generally occur somewhat past the separation line.

In contrast with the 12° calculations, the plot of $\tan^2\beta$ vs ϕ for $\alpha = 8°$ does not indicate separation (Fig. 10). Instead, the curve goes smoothly to zero at $\phi = 180°$.

Detailed flow visualization studies have been performed by Feldhuhn et al.,[35–37] Guffroy et al.,[38] and by George.[39] In particular, these investigators have shown that beyond an incidence somewhat greater than the cone angle, the position of the separation line becomes independent of the angle of attack. Furthermore, at these high angles of incidence, the inviscid flow shows a Reynolds number variation on the leeward meridian, but remains conical near the windward meridian. In the attached flow region, therefore, the similarity assumptions remain valid, and the present theory should adequately predict the separation line location if the experimental pressure distribution is used.

VI. Conclusions

The comparisons of the present theory with data demonstrate the basic validity of the similarity form of the conical boundary-layer equations. In particular, the experimental heat transfer $\dot{q}/\dot{q}_{\alpha=0}$ measured by Tracy was reproduced to within 6% over most of the cone surface, with calculations based on the experimental pressure distributions. Somewhat higher errors occur near the leeward symmetry plane. However, in this region the boundary-layer equations are inadequate, as demonstrated by their inability to predict the sharp growth in boundary-layer thickness observed experimentally near $\phi = \pi$. This inadequacy is also apparent in the inability of the present method to provide a smooth solution near $\phi = \pi$ (except at very small angles of attack), despite the calculation of the previous fluid history.

The use of the experimental pressure distribution permits a precise evaluation of the boundary-layer theory. However, for engineering calculations the experimental pressure distribution is frequently unknown, and it is necessary to solve the inviscid flow as well. Of the several methods investigated for predicting the pressure distribution, the numerical method of Moretti is the most satisfactory, but it somewhat under-predicts the experimental pressures. Stone's perturbation solution generally predicts pressures that are lower still. Finally, the use of Stone's solution for the radial velocity component, together with the assumption of isentropic flow at the surface, is shown to give relatively poor results at high Mach numbers.

The present boundary-layer calculations have been shown to predict surface flow angles which are in good agreement with experimental data, up to the separation point. The ability to predict the location of the separation point depends to a large extent on the validity of the pressure distribution used. Good accuracy is possible where the experimental pressures are available.

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